Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021–8928(99)00050-7
J. Appl. Maths Mechs, Vol. 63, No. 3, pp. 387–396 1999

0021–8928/99/$—see front matter

ASYMPTOTIC BEHAVIOUR AND LEVEL-CURVE STRUCTURE IN PLANE SUBSONIC POTENTIAL FLOWS†

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(*Received* 20 *January* 1998)

The problem of constructing asymptotic forms at infinity and the problem of determining the structure of the isoclines and isobars are considered for uniform plane subsonic potential flow, horizontal at infinity, around a large class of bodies. It is shown that these problems are intimately related. In fact, the construction of a solution in the neighbourhood of the point at infinity (PAI) reduces to (i) selecting a "correct" transformation of the physical plane (PP) onto an auxiliary plane (AP), under which the PAI of the PP goes into the origin of the AP and the gas dynamic equations at the origin of the AP reduce to a Cauchy–Riemann system; (ii) finding the number of isoclines that pass through the PAI and determining the inclinations of these isoclines. With this approach, the construction of asymptotic laws and the investigation of the level curve structure in the neighbourhood of the PAI have much in common with the analogous problem in the neighbourhood of an arbitrary point of the flow at a finite distance from the body. Asymptotic forms are constructed for two cases: lift-creating flow and symmetric flow around a body. The constant factors occurring in the asymptotic formulae are expressed in terms of the physical or geometrical parameters of the problems under consideration.

It will be shown below that for lift-creating flow around a body the isocline passing through the point at infinity (PAI) is perpendicular to the free-stream velocity vector, as follows from the integral law of conservation of mass. Use of the integral law of conservation for the vertical component of the momentum leads to a well-known relationship expressing the lift in terms of circulation (Zhukovski's formula); this in turn uniquely defines the gradients of the angle of inclination of the velocity vector and the pressure, which are mutually perpendicular, and whose moduli are expressed in terms of circulation or lift. This in fact completely defines the asymptotic behaviour at infinity. These results agree, of course, with those obtained previously [1], but the method of solution proposed here is less cumbersome and clearer; moreover, its extension to the case of flow around symmetric bodies requires mathematical tools that do not go beyond the investigation of solutions of the Cauchy–Riemann equations and an analysis of the level curves.

For the circulation-free flow in particular for symmetric flow around a body, there will be more than one isocline through the PAI, as has been pointed out before [1, 2], but the question of the number of isoclines and how it depends on the body geometry remains open. In this paper it will be proved that in symmetric flow around "bell-shaped" bodies (to be defined below), in particular convex bodies, there are no bifurcation points of isoclines and isobars in the physical plane (PP), including, in addition to the earlier results [3], the axis of symmetry. For the larger class of single-vertex bodies, the absence of bifurcation points in the PP will be proved only for null isoclines. In both cases it will be shown that through the origin of the auxiliary plane (AP), corresponding to the PAI of the PP, exactly two curves of zero inclination of the velocity vector pass, one of which, by assumption, is the axis of symmetry. These problems were solved using the method of level curves [3–10] which, under certain conditions, enables one to establish the dependence of the isocline and isobar structure in the flow region on the body geometry. The subsequent analysis of the PAI as a bifurcation point of the type indicated will enable us to evaluate the second derivatives of the gas dynamic parameters at that point, apart from a certain constant factor; as a corollary, it will be possible to determine the quadratic parts of the Taylor series for each gas dynamic parameter, which yield the desired asymptotic form when the linear parts are equal to zero. The physical meaning of the constant factor will also be established. It will be shown that this factor may be expressed directly in terms of the so-called displacement area of a streamline sufficiently far from the body. This area is bounded by a streamline of the flow and its asymptote (a streamline of the initial, unperturbed flow) over its whole extent. This simultaneously shows that, irrespective of quadratic decay of the perturbations as the distance from the body is increased, the displacement area,

†*Prikl. Mat. Mekh.* Vol. 63, No. 3, pp. 403–414, 1999.

which may be regarded as a certain integral parameter of the perturbation introduced into the flow by inserting the body, tends asymptotically to a finite non-zero limit as the distance of the streamline from the body increases.

The results presented below yield one of the first examples in which the method of level curves is used not only to prove the existence or non-existence of various flow regimes, but also to describe flows in certain characteristic regions, in this case—at a considerable distance from the body.

1. We will consider the plane subsonic potential flow, uniform and horizontal at infinity, of an ideal (inviscid and non-heat conducting) gas about a finite closed body. The condition for the flow to be subsonic presupposes that the generatrix of the body is sufficiently smooth; in particular, that there be no convex break points on the generatrixes between the front and rear stagnation points. The flow is described by the following system of equations

$$(u^2-c^2)u_x+uvu_y+uvv_x+(v^2-c^2)v_y=0, \quad u_y-v_x=0 \tag{1.1}$$

Throughout, x and y are Cartesian coordinates, u and v are the components of the velocity vector, q_1 and θ are the modulus and angle of inclination of the velocity vector, p, ρ and c are the pressure, density and speed of sound, which are known functions of q, $M = q/c$ is the Mach number and $k^2 = (1-M_\infty^2)$, the index ∞ denoting the parameters at the PAI.

The boundary conditions for system (1.1) are the impermeability condition at the body surface and the equalities $v = 0$ and $u = q_\infty$ at infinity.

To construct an asymptotic solution of system (1.1) at the PAI, we consider a transformation of the PP (x, y coordinates) onto an AP with Cartesian coordinates α, β

$$\alpha=x(x^2+k^2y^2)^{-1}, \quad \beta=ky(x^2+k^2y^2)^{-1} \tag{1.2}$$

This transformation transfers of the PAI of the PP to the origin of the AP, and system (1.1) becomes a rather cumbersome homogeneous quasi-linear system which, if we define a new function $w = k(u - q_\infty)$, is approximated in the neighbourhood of the origin of the AP by the following Cauchy–Riemann system

$$w_\alpha-v_\beta=0, \quad w_\beta+v_\alpha=0 \tag{1.3}$$

At the point $\alpha = \beta = 0$ itself (the PAI of the PP), system (1.3) is completely equivalent to the original system (1.1). In polar coordinates ε, ω, system (1.3) becomes

$$w_\varepsilon-\varepsilon^{-1}v_\omega=0, \quad \varepsilon^{-1}w_\omega+v_\varepsilon=0$$
$$\text{where } \varepsilon^2=\alpha^2+\beta^2, \ \tan\omega=\beta/\alpha. \tag{1.4}$$

Of all possible transformations which transfer the PAI of the PP to the origin of the AP, the characteristic property of that defined by (1.2) is that it uniquely defines the gradients of the gas dynamic parameters at the origin of the AP. Thus, for example, a transformation which differs from (1.2) solely in that k is replaced by unity leads to a situation in which the gradients of the gas dynamic parameters at the origin of the new AP may not be single-valued but rather depend on the direction of approach to the origin.

The fact that the subsonic flow under consideration is described in the neighbourhood of the origin of the AP by a homogeneous elliptic equation implies [4, 11] that in that neighbourhood the components of the velocity vector have a monotonicity property: each component is monotonic along a level curve of the other component. This is obviously true for the functions p and θ also. Hence, in particular, there can be no closed isoclines enclosing the origin of the AP, from which it follows, in turn, that there is at least one null isocline passing through the origin of the AP and joining it with the body in the flow.

The availability of system (1.3), which asymptotically describes the gas flow in the neighbourhood of the origin of the AP, enables us to reduce the problem of constructing asymptotic forms at infinity to an analysis of the point $\alpha = \beta = 0$, either as a regular point through which only one isocline passes, or as a bifurcation point through which at last two isoclines pass. The next sections will be devoted to calculating the first or second derivatives of the gas dynamic parameters at the origin of the AP, using the Cauchy–Riemann equations (1.3), the integral laws of conservation of mass and momentum and the method of level curves, which will enable us to establish the influence of the body geometry on the isocline and isobar structure in the flow field, including the structure at infinity.

There are well-known examples in gas dynamics of a local transformation of quasi-linear elliptic equations into a Cauchy–Riemann system, following which particular solutions of the system are used to analyse the local properties of the flow. We are referring to flows in the neighbourhood of a concave break point on the body in the flow and near a stagnation point [12], to the interesting property of a flow in the neighbourhood of a cusp, with the concomitant formation of an attached shock wave beyond which the flow becomes subsonic [13–15]—the analysis of this situation led in its time to the concept of the Crocco point on the shock polar, and, finally, to the flow structure in the neighbourhood of a bifurcation point of isobars and isoclines. These results and the results of the present paper, to be discussed below, have much in common and differ only in the boundary conditions in the neighbourhood of the point being investigated, also differing, consequently, in the constants occurring in the particular solutions of the Cauchy–Riemann equations.

2. Let us consider the situation in which there is only one null isocline passing through the origin of the AP, which corresponds to the PAI of the PP. In that case the gradients of the velocity vector components—which, as is evident from (1.3), are mutually perpendicular at the origin of the AP—cannot vanish at that point. In addition, system (1.3) enables us to express the components of one of the gradients in terms of the components of the other and to reduce the problem of constructing asymptotic forms at infinity to that of finding only one of the gradients—say, to fix our ideas, that of the function w, after which it is easy to find the gradients of the other gas-dynamic parameters.

To find the desired values of the derivatives w_α and w_β at the origin of the AP, we will use the integral laws of conservation of mass and the vertical component of the momentum, for which we consider a circle of infinitesimal radius ε about the origin of the AP. In the PP, this circle corresponds to an ellipse with infinitely large semi-axes. The increments dx and dy corresponding to a displacement along the boundary of the ellipse, in the counterclockwise sense, are expressed in terms of the increments $d\alpha$, $d\beta$ and $d\omega$ as follows:

$$dx = \frac{d\alpha}{\varepsilon^2} = -\frac{\sin\omega}{\varepsilon}d\omega, \quad dy = \frac{d\beta}{k\varepsilon^2} = \frac{\cos\omega}{k\varepsilon}d\omega$$

Taking these relations into consideration, we can express the integral law of conservation of mass in terms of the following integral around the circle of radius ε in the AP

$$Fm = \oint \rho(udy - vdx) = \oint \rho(k^{-1}u\cos\omega + v\sin\omega)\frac{d\omega}{\varepsilon} = 0 \tag{2.1}$$

The gas dynamic parameters on the circle are expressed in terms of their values at the origin of the AP (at infinity in the PP) and in terms of their directional derivatives, also evaluated at the origin of the AP. The directional derivative of each of the gas-dynamic parameters is the projection of its gradient onto the relevant direction. Finally, by the definition of the derivative, the values of each of the gas-dynamic parameters on the circle may be written to within $\varepsilon^{1+\delta}$, $\delta > 0$, as

$$\begin{aligned} k^{-1}w &= u - q_\infty = k^{-1}(w_\alpha\cos\omega + w_\beta\sin\omega)\varepsilon \\ v &= (-w_\beta\cos\omega + w_\alpha\sin\omega)\varepsilon \\ p - p_\infty &= c_\infty^2(\rho - \rho_\infty) = -\rho_\infty q_\infty(u - q_\infty) \end{aligned} \tag{2.2}$$

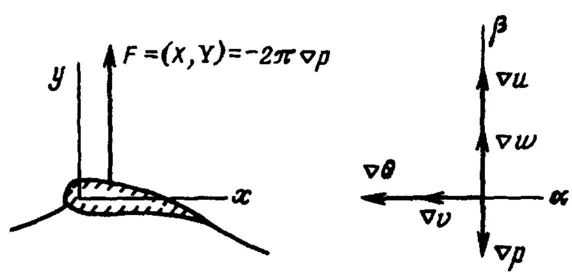

Fig. 1.

Substituting these values into (2.1) and letting ε tend to zero, we find that $Fm = 2\pi\rho_\infty w_\alpha = 0$. It follows that w_α, and therefore also v_β, vanish, the vector ∇w is perpendicular to the vector of the free stream, and the vector $\nabla \upsilon$ is obtained by rotating ∇w through $\pi/2$ in the counterclockwise direction.

Finding analogous expressions for both components of the integral law of conservation of the momentum vector and for the value of the circulation Γ, using Eqs (1.3) and the fact, just proved, that the derivative w_α vanishes at the origin of the AP, we obtain the following expressions for the components of the force vector acting on the body, and expressions for the gradients ∇w and $\nabla \upsilon$ evaluated at the origin of the AP

$$X = 0,\ \ Y = -\rho_\infty q_\infty \Gamma,\ \ 2\pi\rho_\infty q_\infty k^{-1}\nabla w = (0, Y)$$

$$2\pi\rho_\infty q_\infty k^{-1}\nabla v = (-Y, 0) \tag{2.3}$$

The vector equalities in (2.3), together with the obvious equalities

$$k\nabla p = kc_\infty^2 \nabla\rho = -\rho_\infty q_\infty \nabla w,\ \ q_\infty \nabla\theta = \nabla v$$

yield exhaustive information on the behaviour of the gas-dynamic parameters in a small neighbourhood of the origin of the AP, since the linear part of the Taylor series for any gas-dynamic parameter near the origin of the AP is equal to the scalar product of the corresponding gradient and the vector (α, β).

The left-hand part of Fig. 1 shows the body in the flow and the coordinate axes for the PP; the coordinate axes of the AP are shown on the right. This simultaneous use of the PP and AP yields a graphical representation of the asymptotic forms (i.e. the gradients of the gas-dynamic parameters in the AP) and their dependence on the lift. In particular, Fig. 1 illustrates a vector equality which most briefly reflects how the vector of the forces acting on the body depends on the behaviour of the gas-dynamic parameters at the origin of the AP, as well as a corollary of the integral law of conservation of mass

$$p_\alpha = 0,\ \ F = (X, Y) = -2\pi\nabla p$$

In the AP, at a considerable distance from the body, the gas-dynamic parameters are given, apart from higher-order infinitesimals, by the expressions

$$u - q_\infty = \frac{Y}{2\pi\rho_\infty q_\infty}\frac{ky}{z^2} = -\frac{\Gamma}{2\pi}\frac{ky}{z^2}$$

$$v = -\frac{Y}{2\pi\rho_\infty q_\infty}\frac{kx}{z^2} = \frac{\Gamma}{2\pi}\frac{kx}{z^2},\ \ z^2 = x^2 + k^2 y^2$$

which are the well-known Gilbarg and Finn asymptotic formulae [1] (see also [2, 16]). Nevertheless, the method proposed in this paper is of independent interest, since it may be extended to certain symmetric flows, for which several new results are presented below, and possibly also, relying on [10, 11], to certain axially symmetric flows. At the same time, unlike the earlier method used in [1], it is fairly simple and intuitive.

Remark. If, adjacent to the body, a local supersonic zone exists without density jumps, the flow at infinity remains subsonic and irrotational, so that all the results obtained above remain valid. The situation is otherwise if there are shock waves in the local supersonic zone. In that case, the flow in the wake behind the body, formed by streamlines having previously crossed the shock waves, is no longer irrotational and the entropy in it is higher than in the rest of the flow. As a result, the expressions for the components of the force vector acting on the body, and the expressions for the gas-dynamic parameters at considerable distances from the body, differ from those worked out above. In particular, as is well known, the horizontal component of the force vector will then be positive.

3. Let us consider flows corresponding to the case in which there are at least two isoclines passing through the origin of the AP. That this happens in flows around bodies with zero lift and circulation has already been observed [1, 2], but no consideration has been given to the question of the number of isoclines intersecting at the point and the angle of inclination of any of these isoclines. But these are precisely the parameters that determine the form of the asymptotic laws, so that it seems desirable to single out certain classes of bodies for which these parameters can be determined in a unique fashion.

Definition 1. A symmetric body will be called bell-shaped if, in symmetric flow around the body, as one moves along its upper generatrix from the front stagnation point to the rear one, the angle of inclination of the velocity vector first increases monotonically (not necessarily *strictly* monotonically—the same remark will apply throughout this paper) from zero to some positive value, and then decreases monotonically to some negative value, subsequently returning monotonically to zero.

The class of bodies thus defined is quite large. In particular, it contains the class of convex bodies, since the increase of the angle of inclination of the velocity vector on the generatrixes of such bodies is concentrated at the front and rear stagnation points.

Theorem 1. In symmetric subsonic flow about a bell-shaped body, the flow being uniform and horizontal at infinity, there is only one isocline and one isobar passing through every point of the flow region outside the body, including points on the axis of symmetry; in other words, there are no bifurcation points in the flow region. At the same time, the origin of the AP, which corresponds to the PAI of the PP, is a bifurcation point through which two isoclines and two isobars pass; moreover, by the symmetry of the flow, one of the lines $\theta = 0$ is an axis of symmetry.

Proof. We recall the expressions for the derivatives along the lines p = const and θ = const, respectively [4–6]

$$\rho q^2\theta_l = -p_n(1 - M^2 \sin^2 \mu), \quad (1 - M^2)p_l = \theta_n \rho q^2(1 - M^2 \sin^2 \eta)$$

where expressions with the subscripts l and n denote derivatives along the curves and along the normals to the curves, respectively, and μ and η are the angles between the lines p = const and θ = const and the velocity vector.

In view of the possibility that the curve p = const (θ = const) may bifurcate, we henceforth understand an isobar (isocline) to be a curve p = const (θ = const) such that, moving along it, one maintains contact with a selected region of higher or lower p (or θ) values than on the curve itself. In that case, when moving along an isobar (isocline) and passing a possible bifurcation point, one chooses the left-most or right-most branch of the curve p = const (θ = const), depending on the region chosen. Unless otherwise stated, it will be assumed that, when moving along a given isobar or isocline, one must maintain contact with the region on the left.

These definitions, and the expressions presented above for the derivatives evaluated along the curves p = const and θ = const, implying that the angle of inclination of the velocity vector in subsonic flows varies monotonically along an isobar, and correspondingly the pressure varies monotonically along an isocline. These properties provide the basis for the proof of the theorem, which reduces to considering two possible situations.

Suppose a bifurcation point exists in the flow field outside the axis of symmetry. Four or a larger even number of isoclines should issue from this point. These isoclines cannot break off in the flow region and cannot close upon one another. They may either reach the body surface, including possible stagnation points on the body, or go off to infinity. According to the definition of a bell-shaped body, a specific angle $\theta = \theta^* \neq 0$ of the isoclines under consideration at the generatrix, including at the front and rear stagnation points (we are considering the upper half of the body), will correspond to at most two sections, each of which may be of zero length. Only one isocline leaving the assumed bifurcation point may reach each of these sections, for otherwise there would be a closed isocline. If $\theta^* = 0$, only one isocline may go off to infinity—possibly reaching the axis of symmetry—and one more may reach the generatrix of the body. In other words, for any value θ^*, at least two isoclines issuing from the presumed bifurcation point can neither reach the body surface nor go off to infinity.

Now let us suppose that the bifurcation point lies on the axis of symmetry. This case remained unconsidered in the proof that, in flow around a convex body, there are no bifurcation points outside the axis of symmetry [3]. It leads to a situation which, it would seem, does not contradict the properties of isoclines and isobars. Figure 2 illustrates a possible bell-shaped body *acb* and a presumed bifurcation point *d*, from which an isocline *dc* and two isobars *dg* and *df* go up to the body. Segments of the axis of symmetry are also isoclines. In accordance with this scheme, when one is moving along the axis of symmetry from the left, from infinity to the point *d* and then along *dc*, the pressure falls, while along *da* it rises. Along the isobar *df* the value of θ increases, while along *dg* it decreases. The case in which two or more isoclines go from *d* to the upper half of the flow is of no interest, since it is readily disproved

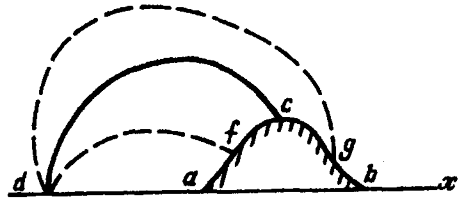

Fig. 2.

by an analysis of isobars and isoclines. That analysis, however, is not enough for the scheme in Fig. 2: here, as it turns out, streamline analysis is also necessary. Indeed, inside the region *dcfa* in Fig. 2, $\theta > 0$, while outside it $\theta \leqslant 0$. Consequently, the inequality $\theta < 0$ holds along the whole of any streamline passing above the isocline *dc*. But it follows from the boundary conditions that y has the same values at $x = \pm\infty$ on any streamline. This contradiction completes the proof that there are no bifurcation points in the flow region, at the same time showing that at least one null isocline, besides the axis of symmetry, goes off from the upper half of the body to infinity. But the existence of two or more such isoclines implies the existence of isoclines coming from infinity, reaching the body surface or the axis of symmetry, and again going off to infinity, which is not compatible with the monotonic variation of the pressure along an isocline. This proves the theorem.

As far as our construction of asymptotic forms below is concerned, only the second part of the theorem is important. We may therefore expect that relaxing the requirement of no bifurcation points in the flow region will enable us to enlarge the class of symmetric bodies for which two isoclines, one of them an axis of symmetry, pass through the origin of the AP.

Definition 2. A symmetric body will be called a single-vertex body if its generatrix can be divided into three parts: a straight segment of maximum cross-section (SMC), adjoining which from the left, upstream, are a front part over which $\theta \geqslant 0$ and, downstream, a rear part over which $\theta \leqslant 0$. The front and rear parts, in turn, may, by definition, contain internal straight segments with $\theta = 0$, adjoining which on either side are side sections with the same signs of θ. Such straight segments with $\theta = 0$ will be called inflection segments, by analogy with the elementary geometric concept of points of inflection on curves. Two types of inflection segments must be distinguished. In the first type the angle θ reaches a local boundary extremum relative to the flow region. No null isoclines issue from such an inflection segment into the flow region, and they will almost never be used in what follows. From an inflection segment of the second type, null isoclines may issue into the flow region; motivated by a certain analogy with saddle points, we will call such inflection segment "saddle inflection segments" (SIS). Of course, each of the straight segments indicated above may be of zero length, i.e. degenerate into a point.

This definition includes a fairly large class of bodies. It excludes from consideration only bodies for which downstream motion along the generatrix may involve a transition from negative values of θ to positive values. In particular, a system of convex bodies arranged successively along the axis of symmetry is not a single-vertex body.

We will now describe some properties of SMC and SIS on a single-vertex body.

Consider an arbitrary curve in the flow region, in a small neighbourhood of the SMC, which begins at the front part, where $\theta > 0$, and ends at the rear part, where $\theta < 0$. Consequently, the equality $\theta = 0$ occurs on this curve an odd number of times. This in turn implies that an odd number of null isoclines leave the SMC and go into the flow region. Let us assume that there are at least three such isoclines, numbered downstream. Consider two isobars issuing from the SMC, the first of which is between the first and second isoclines, and the second between the penultimate and the last. The first of these isobars goes into the region $\theta < 0$, and if one moves along it from the body, θ decreases monotonically. Hence the first isobar must reach the rear part. The second isobar goes into the region $\theta > 0$, and if one moves along it from the body θ increases monotonically, so that the isobar must reach the front part. In other words, these isobars must intersect; but this is impossible, as θ has different signs on each. Consequently, only one null isocline can go from the SMC into the flow region.

It can be shown in exactly the same way that two null isoclines go from a SIS into the flow region.

An important property of each SIS in the front part is that it has at least one boundary point in common with the region $\theta < 0$, which in turn has a boundary point in common with the whole front part or a segment of it. This follows, for example, from the fact that the angle θ evaluated along an isobar issuing from any point of a SIS between the null isoclines, is negative outside the SIS. This carries over in a natural way to an SIS in the rear part.

In what follows it will be logical to call points at which null isoclines leave the body for the flow region $i^+(i^-)$ points if, when one moves along the isoclines away from the body, the left normal is directed into the region $\theta > 0$ ($\theta < 0$). In addition, we introduce a (downstream) enumeration of the points at which null isoclines leave the body surface for the upper half-plane; clearly, the odd numbers in this enumeration label the isoclines issuing from i^+ points.

Theorem 2. In symmetric subsonic flow about a single-vertex body, the flow being uniform and horizontal at infinity, there are no bifurcation points of null isoclines in the flow region and the entire structure of null isoclines is determined by an isocline going from an i^+ point on the SMC. If there is no SIS on the body, this isocline goes off to infinity; otherwise it "visits" all the SIS successively, reaching each of them at an i^- point and leaving it at an i^+ point. In the first stage, the isocline, having issued from the SMC at the nth point, will reach the $(n-1)$th or $(n+1)$th point, and then the two numbers corresponding to the isocline will be excluded from consideration, the enumeration of the points will shift accordingly, and the entire procedure will be repeated. It is clear from this scheme that the last point at which an isocline leaves the body before going off to infinity will be either the first or the last point of exit of a null isocline (in the initial enumeration). Finally, it follows from this structure of the null isoclines that exactly two curves $\theta = 0$ intersect at the origin of the AP, one of them being the axis of symmetry.

Proof. The fairly detailed statement of the theorem makes the proof almost obvious. We merely observe that if there is at least one SIS on the body, analysis of the isobars issuing from points of the SIS shows that in the neighbourhood of the SMC one can single out a region, bounded by an isobar which begins and ends on the body, along which $\theta \neq 0$ outside the body, and part of the generatrix of the body on which there are only two exit points of null isoclines, i^+ on the SMC and i^- on one of the SIS. Therefore, an isocline, having left from an i^+ point on the SMC, may arrive only at an i^- point on the last-mentioned SIS and moreover without any bifurcation points in between. Successive repetition of these arguments completes the proof.

To illustrate, Fig. 3 represents a single-vertex body containing three inflection segments and showing one of the possible versions of a null isocline structure, on the assumption that each of the inflection segments is an SIS. The solid curves denote null isoclines, of which only the left-most one goes off to infinity. The dashed curves denote isobars, one of which is that occurring in the first part of the proof.

Note that the theorem yields a finite number of possible versions of null isocline structure, but does not indicate which of the inflection segments is an SIS. The significance of the theorem is the unambiguous answer it gives to the question of the structure of the bifurcation point at the origin of the AP, which is necessary to construct the asymptotic form at infinity.

Theorem 3. In symmetric subsonic flow around a single-vertex (in particular, a bell-shaped or convex) body, the flow pattern in the neighbourhood of the origin of the AP is uniquely defined by the values of the second derivatives of the gas dynamic parameters at the origin of the AP; these derivatives may be written, using an as yet undetermined positive factor a^2, as follows:

$$w_{\alpha\beta} = v_{\alpha\alpha} = v_{\beta\beta} = 0, \quad w_{\alpha\alpha} = -w_{\beta\beta} = v_{\alpha\beta} = -2q_\infty a^2 \tag{3.1}$$

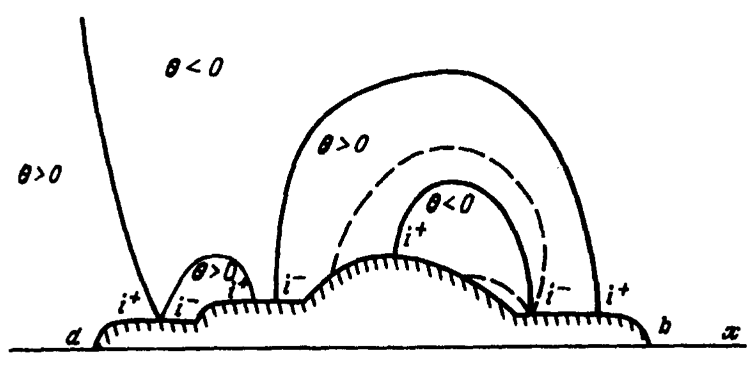

Fig. 3.

As a corollary, the functions w and v, defined in this case by the quadratic parts of the corresponding Taylor series, have the following form in the neighbourhood of the origin of the AP, apart from quantities of the order of $\varepsilon^{2+\delta}$, $\delta > 0$

$$w = k(u - q_\infty) = -q_\infty a^2(\alpha^2 - \beta^2), \quad v = -2q_\infty a^2\alpha\beta$$

Accordingly, in the PP, at considerable distances from the body, the functions w and v may be expressed, apart from quantities of the order of $(x^2 + k^2y^2)^{-1-\delta}$, $\delta > 0$, as follows:

$$w = k(u - q_\infty) = -q_\infty a^2 \frac{x^2 - k^2y^2}{(x^2 + k^2y^2)^2}, \quad v = -q_\infty a^2 \frac{2kxy}{(x^2 + k^2y^2)^2} \tag{3.2}$$

Proof. It follows from the above results that all the first derivatives of the gas-dynamic parameters vanish at the origin. Consequently, the behaviour of these functions in the neighbourhood of the origin of the AP is determined by their second derivatives, provided that they do not vanish simultaneously. To evaluate these derivatives at $\alpha = \beta = 0$ we return to the original system (1.1) which, through the use of (1.2), was transformed into a homogeneous quasi-linear system in the derivatives w_α, w_β, υ_α, υ_β. The advantage of the change of variables (1.2) is that in the transformed system of equations

$$a_{11}w_\alpha + a_{12}w_\beta + b_{11}\upsilon_\alpha + b_{12}\upsilon_\beta = 0, \quad a_{21}w_\alpha + a_{22}w_\beta + b_{21}\upsilon_\alpha + b_{22}\upsilon_\beta = 0$$

the variable coefficients take the following form at the origin of the AP

$$a_{12} = b_{11} = a_{21} = b_{22} = 0, \quad a_{11} = -b_{12} = a_{22} = b_{21} = 1$$

so that the system reduces at the origin of the AP to the Cauchy–Riemann system (1.3).

To determine the second derivatives at the origin of the AP, we construct two systems of two equations, one of which is obtained by differentiating the above system with respect to α, the other by differentiating it with respect to β. Taking into consideration that the first derivatives vanish at the origin of the AP, we obtain the following four equations, valid at the origin of the AP

$$w_{\alpha\alpha} - \upsilon_{\alpha\beta} = 0, \quad w_{\alpha\beta} + \upsilon_{\alpha\alpha} = 0, \quad w_{\alpha\beta} - \upsilon_{\beta\beta} = 0, \quad w_{\beta\beta} + \upsilon_{\alpha\beta} = 0$$

To these equations we must add the equality $\upsilon_{\alpha\alpha} = 0$, which follows from the fact that $\upsilon = 0$ on the axis of symmetry, as well as the inequality $\upsilon_{\alpha\beta} \leqslant 0$, which reflects a result of Theorem 2—the absence of bifurcation points of isoclines in the PP, including on the axis of symmetry. This last inequality is conveniently replaced by an equality $\upsilon_{\alpha\beta} = -2q_\infty a^2$, where a^2 is an as yet undetermined constant.

Thus, the six equalities written out above prove the validity of the expressions for the second derivatives in the statement of Theorem 3. Analysis of the quadratic parts of the Taylor series for w and υ in the neighbourhood of the point $\alpha = \beta = 0$, assuming $a \neq 0$, shows that the two curves $w = 0$ and the two curves $\upsilon = 0$ intersect at that point, in agreement with Theorem 2. Now, to complete the proof of Theorem 3, we must show that the constant a cannot vanish, so that, consequently, the second derivatives cannot all vanish simultaneously at the origin of the AP.

Indeed, if $a = 0$, then completely analogous calculations yield the values of the third and, if necessary, higher-order derivatives at $\alpha = \beta = 0$, and an analysis of these derivatives shows that three or more curves $\upsilon = 0$ intersect at that point, contradicting Theorem 2. This completes the proof of Theorem 3.

It is worth observing here that an approach based on unrestricted expansion of the neighbourhood of the point under consideration is widely used in fluid mechanics when investigating certain local singularities (a fracture in a wall [12], the attached shock wave of a weak subsonic family [13–15], the bifurcation of level curves, the discontinuity in the curvature of a wall, etc.). In particular, when this approach is taken, the boundary conditions valid in the neighbourhood of the point carry over to unbounded straight-line segments. The solution obtained generally yields an exact characterization of the singularity in question. Applied to our case, this approach postulates that a solution of the Cauchy–Riemann system (1.4) is constructed in the upper half-plane of the AP, on the assumption that $w = \upsilon = 0$ at the origin of the AP, and $v = 0$ when $\beta = 0$. It is easy to verify that this condition is satisfied by the following family of solutions with a non-zero factor A and positive integer n

$$w = -q_\infty A\varepsilon^n \cos n\omega, \quad v = -q_\infty A\varepsilon^n \sin n\omega$$

The parameter n is determined on the basis of the results of Theorem 2, according to which exactly one curve

$\upsilon = 0$ can reach the origin of the AP from the upper half-plane of the AP. Hence we find that $n = 2$. It is further shown in Theorem 2 that there the curves $\upsilon = 0$ have no bifurcation points within the flow region. Consequently, the derivative υ_β does not vanish on the axis of symmetry of the AP, except at the origin, whence we conclude, taking the body geometry into account, that $\upsilon_\omega < 0$ on the entire line $\beta = 0$, except for the origin of the AP. As a result, we find that $A > 0$. The agreement of these results with those of Theorem 3 indicates that this approach may legitimately be applied to our problem.

Clearly, the factor a^2, which has the dimension of area, depends in some way on the shape of the body. It can also be seen that its value characterizes the integral value of the perturbation caused in the flow by the body. These discussions justify considering the following idea.

Definition 3. Considering flow around a symmetric closed body, we define the displacement area $S(h)$ as the area between the infinite horizontal line $y = h$ and the streamline for which this line is an asymptote. The limit S of the displacement area will be called the displacement area for infinitely large h.

As it turns out, this concept is directly related to the factor a^2 of (3.2).

Theorem 4. In symmetric subsonic flow about a single vertex body (in particular, bell-shaped or convex bodies), the flow being uniform and horizontal at infinity, the limit of the displacement area is equal to the product of the number π and the positive factor a^2 in the asymptotic forms (3.1) and (3.2), that is, $S = S(\infty) - \pi a^2$. This relationship extends to a larger class of closed bodies, provided that the asymptotic behaviour of the flow at infinity is described by (3.1) and (3.2).

Proof. Let us consider a streamline, sufficiently far from the body, for which the line $y = h$ is an asymptote; that is, $y = h$ on the streamline at $x = \pm\infty$. If $h \gg 1$, the value of the derivative dy/dx on the streamline may be replaced, to within higher order infinitesimals, by $\sin\theta$, which in turn may be replaced to within some error (which can be estimated) by the value of $\sin\theta$ evaluated on the asymptote for the same value of x by using the second relation of (3.2). Integration of this relation—of course, without the factor q_∞—from $-\infty$ to $+\infty$ and estimating all the above-mentioned errors, apart from an unimportant bounded constant b, we obtain an asymptotic expression for $S(h)$ and, in the limit when $h = \infty$, the required expression for the limit S of the displacement area

$$S(h) = \pi a^2 + bh^{-1}, \quad S = S(\infty) = \pi a^2$$

Remark. In the case of a single-vertex body, owing to the presence of a local supersonic zone without density jumps, the flow remains subsonic and irrotational at infinity. Further analysis of a fictitious boundary made up of the subsonic parts of the body and the sonic line, using a theorem of Nikol'skii and Taganov [4], shows that the fictitious boundary satisfies the conditions concerning the variation of the angle of inclination of the velocity vector in Definition 2. It follows that in this case, too, only two null isoclines intersect at the origin of the AP, so that Theorems 3 and 4 remain valid.

Note that formulae (3.2) yield an exact solution for the whole flow region in the case of the symmetric flow of an incompressible fluid around a circular cylinder, provided that the radius of the cylinder is equal to the constant a of (3.2). In that case the displacement area for $h = 0$ is exactly equal to the area of the figure around which the flow is taking place, that is, the upper half of the cylinder, and moreover as h increases to infinity this area is doubled: 413b. Note also that there is as yet no answer to the question of whether closed bodies, or finite systems of closed bodies, exist, in the flow around which the PAI is a bifurcation point of at least the third order (at least three curves $\theta = 0$ intersect at the origin of the AP), so that, consequently, the limit of the displacement area vanishes.

I wish to thank Yu. D. Shmyglevskii for his comments at a seminar and during subsequent discussion.
This research was supported financially by the Russian Foundation for Basic Research (97-01-00833).

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Translated by D.L.